Some remarks about Descartes' rule of signs

Alain Albouy, Yanning Fu

Some remarks about Descartes' rule of signs

Alain Albouy[1,2], Yanning Fu[1]

[1] Purple Mountain Observatory
Chinese Academy of Sciences
2 West Beijing road, Nanjing 210008, China
[2] CNRS-UMR8028, IMCCE
Observatoire de Paris
77 avenue Denfert-Rochereau, 75014 Paris, France

albouy@imcce.fr, fyn@pmo.ac.cn

Abstract. What can we deduce about the roots of a real polynomial in one variable by simply considering the signs of its coefficients? On one hand, we give a complete answer concerning the positive roots, by proposing a statement of Descartes' rule of signs which strengthens the available ones while remaining as elementary and concise as the original. On the other hand, we provide new kinds of restrictions on the combined numbers of positive and negative roots.

1. Coefficients and positive roots.

It is convenient to consider that an analytic function in one variable has a *multiset* of roots. A multiset is defined just as a set, except that identical elements are allowed. The *cardinality* of a multiset is the number of its elements. The *multiplicity* of an element is the cardinality of the sub-multiset of the elements identical to it.

Consider expressions of the type

$$Y = a_0 x^{\alpha_0} + a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n}, \tag{1}$$

with real exponents $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ and non-zero real coefficients $a_0, \ldots, a_n$. The *sequence of signs* of an expression (1) is the ordered list $\sigma_0, \ldots, \sigma_n$ of the signs $\sigma_i = |a_i|^{-1} a_i$. A *variation* occurs in such a sequence when $\sigma_i = -\sigma_{i+1}$. For example the sequence $1, -1, 1, 1$ has two variations.

We say that two expressions of type (1) belong to the same *class* if they have same number of terms, same list of exponents and same sequence of signs.

Proposition 1. Let us fix arbitrarily a class and a multiset of positive numbers. We call k the number of variations in the sequence of signs associated to the class, and p the cardinality of the multiset. There exists in the given class a multiterm expression (1) with the given multiset of positive roots if and only if $k - p$ is even and non-negative.

This statement gives an "if and only if" form to the classical Descartes' rule of signs. It continues previous works. In 1998, Anderson, Jackson and Sitharam [1] proposed examples of polynomials with any sequence of signs, and with any number p of positive roots, provided that $k - p$ is even and non-negative. In 1999, Grabiner [5] extended this result by giving also examples of polynomials with missing terms (i.e. zero coefficients for some of the intermediate integer powers of x). Our Proposition 1 strengthens this statement, by extending these previous results to real exponents, and by showing that when a given sequence of signs allows p positive roots, it indeed allows any multiset of positive roots of cardinality p. Surprisingly the proof, as presented below, remains very elementary.

2. Proof of the "only if" part.

This proof is well-known and we give it only for completeness. We assume that the given multiset is the multiset of positive roots of Y.

(i) The parity of k decides if the sign of a_0 is or not the sign of a_n. The parity of p decides if the sign of Y is or not the same at the two ends, i.e. at 0^+ and at $+\infty$. Obviously these boundary signs are respectively the signs of a_0 and of a_n. Thus $k - p$ is even.

(ii) Let m and m' be the cardinalities of the respective multisets of positive roots of a function f and of its derivative f'. Then $m \leq m' + 1$.

(iii) One multiplies Y by $x^{-\alpha}$, computes the derivative in x and observes the resulting sequence of signs. The signs corresponding to the exponents $\alpha_i < \alpha$ are changed, while those corresponding to $\alpha_i > \alpha$ are not changed. We can delete any given variation of sign without touching the others by choosing α in some interval of the form $]\alpha_i, \alpha_{i+1}[$.

To conclude, we prove that $p \leq k$ by an induction on k. If $k = 0$ then $p = 0$. If Y has k variations, argument (iii) shows that $(x^{-\alpha}Y)'$ has $k-1$ variations for some choice of α. Argument (ii) gives the required estimate on the respective numbers of roots. QED

These arguments are actually older than usually thought. Argument (ii) was given in 1741 by de Gua and again in 1798 by Lagrange. Argument (iii) and the statement of the "only if" part are given by de Gua in the case of integer exponents, by Laguerre in the case of real exponents[1]. Laguerre is explicit about the parity conclusion (i), which is obvious. If the statement of the "only if" part omits this conclusion, then it remains correct if the roots are counted without multiplicity[2].

3. Proof of the "if" part.

We call $(\sigma_0, \dots, \sigma_n)$ the prescribed sequence of signs (with k variations). We assume that the cardinality p of the given multiset of positive numbers is such that $k-p$ is even and positive. We look for an expression (1) with $\sigma_i a_i > 0$ for all i, such that the given multiset is the multiset of its positive roots. We will be able to find such an expression with the additional constraint that $a_i = a_{i+1}$ when $\sigma_i = \sigma_{i+1}$. We collect the terms in (1) according to this constraint. We set $i_0 = 0$ and call $i_1, i_2, \dots, i_k$ the integers i_j such that $i_{j-1} < i_j$, $\sigma_{i_j-1} \neq \sigma_{i_j}$, $\sigma_{i_k} = \sigma_n$. We call $b_0, b_1, \dots, b_k$ the common values of the coefficients:

$$b_j = a_{i_j} = a_{i_j+1} = \cdots = a_{i_{j+1}-1}.$$

We set

$$\varphi_j(x) = x^{\alpha_{i_j}} + x^{\alpha_{i_j+1}} + \cdots + x^{\alpha_{i_{j+1}-1}}.$$

Now we look for a $(b_0, b_1, \dots, b_k)$ such that $\sigma_0 b_0 > 0$, $b_i b_{i+1} < 0$ and the expression

$$Y = b_0\varphi_0(x) + b_1\varphi_1(x) + \cdots + b_k\varphi_k(x)$$

[1]See [6] pp. 89–92, [10] p. 195, [11]. Jensen [8] noticed the similarity of de Gua's and Laguerre's arguments. Instead of derivation, de Gua used Johan Hudde's operation, which is described as a term by term multiplication of the polynomial and an arithmetic progression. The modern reader will simply compare this with the operation $Y \mapsto x^{-m+1}(x^m Y)'$.

[2]Authors of the 18th century used to count the roots with their multiplicity and to omit the parity conclusion. Incidentally, many authors following Cajori [2] attribute the parity conclusion to Gauss [4] in 1828. This is very strange as firstly, this conclusion is obvious, secondly, Fourier published it in 1820 (see [3] p. 294) and thirdly, it is absent from Gauss' paper.

has the required multiset of positive roots.

We begin with the case where all the roots $x_1, \dots, x_p$ are simple. The column vector $(b_0, \dots, b_k)$ should be in the kernel of the matrix

$$\Phi = \begin{pmatrix} \varphi_0(x_1) & \varphi_1(x_1) & \cdots & \varphi_k(x_1) \\ \varphi_0(x_2) & \varphi_1(x_2) & \cdots & \varphi_k(x_2) \\ & & \cdots & \\ \varphi_0(x_p) & \varphi_1(x_p) & \cdots & \varphi_k(x_p) \end{pmatrix}.$$

This $p \times (k+1)$ matrix has a non-trivial $(b_0, \dots, b_k)$ in its kernel since $p \leq k$.

If $p = k$, the "only if" part shows that the coefficients b_i are non-zero, that their signs alternate and that there is no other root. This makes an $(a_0, \dots, a_n)$ such that Y has the prescribed set of positive roots.

If $1 \leq p < k$, observe that the $p \times p$ submatrix formed by the first p columns of Φ is invertible. If its determinant was zero we could find a non-trivial column vector $(c_0, \dots, c_{p-1})$ in its kernel. But $\sum_0^{p-1} c_i \varphi_i(x)$ would have a root at each x_j. It would have p roots and at most $p-1$ variations, which contradicts what we proved in the "only if" part.

So after setting $b_k(\epsilon) = \sigma_n$ and $b_i(\epsilon) = (-1)^{k-i}\sigma_n\epsilon$, $i = p, \dots, k-1$, there is for any ϵ a unique $(b_0(\epsilon), \dots, b_{p-1}(\epsilon))$ such that the column vector $(b_0(\epsilon), \dots, b_{p-1}(\epsilon), b_p(\epsilon), \dots, b_k(\epsilon))$, regular in ϵ, is in the kernel of Φ.

Call Y_ϵ the expression Y with coefficients $b_i(\epsilon)$. It vanishes at the x_i's. As $k - p$ of the coefficients of Y_0 vanish, the $p + 1$ remaining ones should be non-zero with alternate signs according to the "only if" part, which at the same time establishes that Y_0 cannot have any other positive root. As $k - p$ is even, the signs agree with the prescribed sequence of signs when $\epsilon > 0$.

Consider Y_ϵ / Y_0. This quotient tends to a finite limit at the x_i's, which are zeros of the numerator and the denominator. As $\epsilon \to 0$, it tends uniformly to 1, as shown by usual techniques. Thus Y_ϵ has no other positive root for a sufficiently small ϵ. The problem is solved in the case of simple roots.

In the cases with multiple roots the construction and the proofs are exactly the same, except that Φ has e.g. the row $(\varphi_0'(x_1), \dots, \varphi_k'(x_1))$ if x_1 is a multiple root, the row $(\varphi_0''(x_1), \dots, \varphi_k''(x_1))$ if it is at least a triple root, etc.

If finally $p = 0$, k is even and we can build a Y without root. We choose $b_0(\epsilon) = \sigma_0$, $b_k(\epsilon) = \sigma_0$ and $b_i(\epsilon) = (-1)^i \epsilon \sigma_0$. We choose a sufficiently small ϵ such that there is no positive root. QED

4. Further information about trinomials.

Etymologically a trinomial is simply the sum of three monomials, and there is no restriction on the degree. Trinomials with unprescribed degree were studied early in the history of algebraic equations (see [14], pp. 11 and 24). They are natural objects in the context of Laguerre's extension of Descartes' rule to real exponents, as well as in the context of Khovansky's theory of fewnomials [9], and are consequently the object of recent studies. In 2002, Haas [7], Li, Rojas & Wang [12] proved that the optimal upper bound on the number of roots in the positive quadrant of a system of two trinomials in two variables is five.

We were not able to find the following elegant and elementary formula in any of these old or recent studies.

Proposition 2. A trinomial $ax^\alpha + bx^\beta + cx^\gamma$, $\alpha < \beta < \gamma$, $a > 0$, $c > 0$, $b < 0$, is positive on $]0, \infty[$ if and only if

$$\Big(\frac{a}{\gamma-\beta}\Big)^{\gamma-\beta}\Big(\frac{b}{\alpha-\gamma}\Big)^{\alpha-\gamma}\Big(\frac{c}{\beta-\alpha}\Big)^{\beta-\alpha} > 1.$$

Remark. The discriminant of $ax^\alpha + bx^\beta + cx^\gamma$ is $b^2 - 4ac$ if $(\alpha, \beta, \gamma) = (0, 1, 2)$. It is $-c(4b^3 + 27ca^2)$ if $(\alpha, \beta, \gamma) = (0, 1, 3)$. It is $c^2(-27b^4 + 256ca^3)$ if $(\alpha, \beta, \gamma) = (0, 1, 4)$ and $16ac(4ca - b^2)^2$ if $(\alpha, \beta, \gamma) = (0, 2, 4)$. These formulas are quite familiar. The general formula in the proposition gives in each case the main factor.

The formula is obtained by a straightforward computation, but there occur unexpected simplifications. It is interesting to try to find a shortcut. Consider the trinomial $Y = ax^\alpha + bx^\beta + cx^\gamma$. The explicit expression in the existence proof of the previous section shows that Y has a double root at x_1 if and only if

$$\frac{ax_1^\alpha}{\gamma-\beta} = \frac{bx_1^\beta}{\alpha-\gamma} = \frac{cx_1^\gamma}{\beta-\alpha}.$$

We set

$$A = \frac{a}{\gamma-\beta}, \quad B = \frac{b}{\alpha-\gamma}, \quad C = \frac{c}{\beta-\alpha},$$

which are positive numbers according to the hypotheses of the proposition, and continue the computation by eliminating x_1, which gives the expected condition $A^{\gamma-\beta}B^{\alpha-\gamma}C^{\beta-\alpha} = 1$ for a double root, which is located at

$$x_1 = \Big(\frac{C}{B}\Big)^{1/(\beta-\gamma)} = \Big(\frac{A}{C}\Big)^{1/(\gamma-\alpha)} = \Big(\frac{B}{A}\Big)^{1/(\alpha-\beta)}.$$

5. Combining positive and negative roots.

If we just focus on the positive roots of a given polynomial Y, Proposition 1 tells us that there is nothing we can add to Descartes' rule of signs. All that can be deduced from the sign sequence of Y is the upper bound and the parity of P, the cardinality of the multiset of positive roots. By changing x to $-x$, the same can be said on N, the cardinality of the multiset of negative roots of Y. But let us consider the constraints on (P, N).

Consider a polynomial $Y(x)$ of degree 4 with sign sequence $+, -, -, -, +$. By Descartes' rule $P = 0$ or 2. If we change x to $-x$ the sign sequence becomes $+, +, -, +, +$. Thus $N = 0$ or 2. Grabiner [5] points out that $(P, N) = (0, 2)$ is *impossible* for such a $Y(x)$.

The proof does not require any computation. Because the constant term $Y(0)$ is positive, $P = 0$ implies $Y > 0$ for $x > 0$. But the odd part of Y is negative when $x > 0$. So the even part is positive. For $x < 0$ the odd and the even parts are then positive. We must have $N = 0$. QED.

Consider an expression (1) as in Proposition 1, where the α_i's are non-negative integers, i.e. $Y(x)$ is a polynomial. Given the sequence of signs $\sigma_0, \ldots, \sigma_n$, Grabiner proves[3] that all the (P, N)'s produced as follows are *possible*. One chooses any subset of $\{1, \ldots, n-1\}$ and "erases" the corresponding σ_i's from the sequence of signs. The number of variations in the resulting sequence gives P. One then considers the modified sequence of signs $\sigma_0(-1)^{\alpha_0}, \sigma_1(-1)^{\alpha_1}, \ldots, \sigma_n(-1)^{\alpha_n}$ and erases from it the signs with index in the same subset. The new number of variations gives N.

Let us call this construction of possible (P, N)'s as Grabiner's erasing term rule. Grabiner observes that it gives all the possible (P, N)'s for polynomials up to degree 4, and conjectures that the same is true for higher degrees.

Here is a counterexample. Consider a polynomial of 5th degree with sign sequence $+, +, -, +, +, -$. Here $(P, N) = (3, 0)$ is possible, as shown by

$$(20+37x+18x^2)(1-x)(2-x)(3-x)=120+2x-179x^2+4x^3+71x^4-18x^5.$$

This combination is not obtained by Grabiner's erasing term rule. Indeed, after changing x in $-x$ the sequence becomes $+, -, -, -, +, +$. As $N = 0$ we should erase terms and obtain a sequence without variation. We should

[3]The idea is that small terms can be neglected before applying Descartes' rule. Estimates on how small these terms should be are discussed in [13]. It is interesting to compare these estimates to the first Lemma in [6].

erase either the three $-$, or the three $-$ and the internal $+$. But erasing the corresponding signs in $+,+,-,+,+,-$ gives in both cases only one variation, while $P=3$.

In contrast $(P,N)=(3,0)$ is impossible for the sequence $+,+,-,+,-,-$. Descartes' rule predicts all the impossibilities for degree 5 polynomials without gaps, except this one and its trivial analogous.

To prove this impossibility, let us specify the ordering of sign sequences by associating the first sign to the constant term. We notice that the odd part $\mathcal{O}$ has sequence of signs $+,+,-$. Its derivative of course has the same sequence and consequently a unique positive root r. On $]0,r[$, $\mathcal{O}$ is positive, and for $x>r$, $\mathcal{O}$ is decreasing. The even part $\mathcal{E}$ has sequence of signs $+,-,-$, thus it decreases for $x>0$. Our polynomial $\mathcal{E}+\mathcal{O}$ decreases and has at most 1 root on $]r,+\infty[$. As $N=0$, $\mathcal{E}-\mathcal{O}$ is positive on $]0,+\infty[$. On $]0,r[$, $\mathcal{O}$ and thus $\mathcal{E}+\mathcal{O}$ are positive. There is one positive root on $]r,+\infty[$, thus $P=3$ is impossible. QED

Let us pass to degree 6 polynomials. The exhaustive list of non-Descartes impossibilities is, up to trivial transformations:

$$+,+,-,+,-,+,+ \text{ is incompatible with } (P,N)=(2,0) \text{ or } (4,0)$$

$$+,+,+,+,-,+,+ \text{ is incompatible with } (P,N)=(2,0)$$

$$+,+,-,-,-,-,+ \text{ is incompatible with } (P,N)=(0,4)$$

The argument we gave in Grabiner's example proves the first and second statements.

To prove the third statement, we write a polynomial with $(P,N)=(0,4)$ as

$$p=(c-bx+x^2)(x+x_1)(x+x_2)(x+x_3)(x+x_4),$$

where $x_i>0$, $i=1,\dots,4$,

$$4c>b^2>0. \tag{2}$$

Expanding, $p=a_0+a_1x+\cdots+a_5x^5+x^6$, with in particular

$$a_2=c\beta-b\gamma+\delta<0, \tag{3}$$

$$a_5=-b+\alpha<0, \tag{4}$$

the positive numbers $\alpha,\beta,\gamma,\delta$ being defined as

$$\begin{aligned}\alpha&=x_1+x_2+x_3+x_4\\ \beta&=x_1x_2+x_1x_3+x_2x_3+x_1x_4+x_2x_4+x_3x_4\\ \gamma&=x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4\\ \delta&=x_1x_2x_3x_4\end{aligned} \tag{5}$$

By (2) and (4),

$$\sqrt{c} > b/2 > \alpha/2 > 0, \tag{6}$$

and, by (2) and (3),

$$2\sqrt{c} > b > \frac{c\beta + \delta}{\gamma},$$

which gives

$$\beta\sqrt{c}^2 - 2\gamma\sqrt{c} + \delta < 0. \tag{7}$$

We expand the discriminant $4\gamma^2 - 4\beta\delta$ of this expression using (5). All the terms are positive. Now (7) implies in particular that

$$\sqrt{c} < \frac{\gamma + \sqrt{\gamma^2 - \beta\delta}}{\beta}.$$

Combining with (6) gives $\beta\alpha/2 < \beta\sqrt{c} < \gamma + \sqrt{\gamma^2 - \beta\delta}$ or

$$\alpha\beta/2 - \gamma < \sqrt{\gamma^2 - \beta\delta}. \tag{8}$$

After expanding and canceling, we see that all the terms of $\alpha\beta/2 - \gamma$ are positive. We square both sides of (8). This gives us $\alpha^2\beta - 4\alpha\gamma + 4\delta < 0$. But the left hand side, after expanding and canceling, has only positive terms. This is a contradiction. QED

To check that our list of impossibilities is complete at degree 6, it is enough to consider the possibilities established by Grabiner's erasing term rule and the two following polynomials, which after changing x into $-x$, x into $1/x$ or Y into $-Y$ provide examples for all the other possibilities:

$$(31+11x+x^2)(1-x)(2-x)(3-x)(4-x)=744-1286x+559x^2+25x^3-44x^4+x^5+x^6,$$

$$(9+8x+2x^2)(1-x)(2-x)(3-x)(4-x)=216-258x-37x^2+90x^3-x^4-12x^5+2x^6.$$

These possibilities and impossibilities do not seem to organize themselves in simple classes.

Acknowledgements. We wish to thank Alain Chenciner, David Grabiner and Dieter Schmidt. This work is supported by Chinese Academy of Sciences visiting professorship for senior international scientists grant no. 2009J2-11, and NSFC grant no. 10833001.

References

[1] Bruce Anderson, Jeffrey Jackson, Meera Sitharam, *Descartes' rule of signs revisited*, The American Mathematical Monthly 105 (1998) pp. 447–451

[2] Florian Cajori, *A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity*, Colorado College Publication, Science series 12–7 (1910) pp. 171–215

[3] Joseph Fourier, *Sur l'usage du théorème de Descartes dans la recherche des limites des racines*, Bulletin des sciences par la Société philomatique de Paris (1820) pp. 156–165, 181–187; *œuvres* 2, pp. 291–309, Gauthier-Villars, 1890

[4] Carl Friedrich Gauss, *Beweis eines algebraischen Lehrsatzes*, Journal für die reine und angewandte Mathematik 3 (1828) pp. 1–4; *werke* 3, pp. 67–70, Göttingen, 1866

[5] David J. Grabiner, *Descartes' Rule of Signs: Another Construction*, The American Mathematical Monthly 106 (1999) pp. 854–856

[6] Jean-Paul de Gua de Malves, *Démonstrations de la Régle de Descartes, Pour connoître le nombre des Racines positives & négatives dans les Équations qui n'ont point de Racines imaginaires*, Memoires de Mathématique et de Physique tirés des registres de l'Académie Royale des Sciences (1741) pp. 72–96

[7] Bertrand Haas, *A Simple Counterexample to Kouchnirenko's Conjecture*, Beiträge zur Algebra und Geometrie, Contributions to Algebra and Geometry 43 (2002) pp. 1–8

[8] J.L.W.V. Jensen, *Recherches sur la théorie des équations*, Acta Mathematica 36 (1913) pp. 181–195

[9] A.G. Khovanskiĭ, *Fewnomials*, Translations of mathematical monographs 88, American Math. Society, 1991

[10] Joseph Louis Lagrange, *Traité de la résolution des équations numériques de tous les degrés, avec des Notes sur plusieurs points de la Théorie des*

équations algébriques, Courcier, Paris 1798 and 1808; *œuvres* 8, Gauthier-Villars, 1879

[11] Edmond Laguerre, *Sur la théorie des équations numériques*, Journal de Mathématiques pures et appliquées, s. 3, t. 9 (1883) pp. 99–146; *œuvres* 1, Paris, 1898, Chelsea, New-York, 1972, pp. 3–47

[12] Tien-Yien Li, J. Maurice Rojas, Xiaoshen Wang, *Counting real connected components of trinomial curve intersections and m-nomial hypersurfaces*, Discrete Comput. Geom. 30 (2003) pp. 379–414

[13] Christian Marchal, *Une amélioration commode du critère de Descartes*, Sciences, AFAS, fasc. 4 (2006) pp. 3–6

[14] Jacqueline Stedall, *From Cardano's great art to Lagrange's reflections: filling a gap in the history of algebra*, European Mathematical Society, 2011